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ROCKING ROTATION OF A RIGID DISC IN A HALF-SPACE

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Abstract—An analysis is presented for the determination of the rotational response of a rigid circular disc embedded in a semi-infinite elastic medium under the action of a rocking moment. With the aid of Hankel transforms, a relaxed treatment of the mixed boundary value problem is formulated as dual integral equations. On reduction of the dual integral equations to a Fredholm integral equation which features a closed-form kernel, solutions to the inclusion problem are computed. In addition to providing a unified view of existing solutions for zero and infinite embedments, the present analysis reveals a severe boundary-layer phenomenon which is apt to be of significance to this class of problem in general. As illustrations, numerical results on the load–displacement relation and the response of the embedding medium, as well as the contact load distribution, are included.

1. INTRODUCTION

On the subject of structure–medium interaction, one class of fundamental problems is the determination of the response of a loaded rigid disc which is in contact with an elastic medium. These analyses are relevant to engineering in the context of soil–structure interaction, footing and anchor designs. They are also related to the analysis of indentation processes, stress concentrations, and fracture mechanics of composite materials. For this category of problems, considerable attention has been paid to the case where the disc is resting on the surface of an elastic half-space as in Boussinesq (1885), Reissner (1937), Abramov (1939), Reissner and Sagoci (1944), Harding and Sneddon (1945), Sneddon (1947, 1966), Bycroft (1956), Ufliand (1956), Keer (1967), Spence (1968), and Gladwell (1969). A variety of problems associated with a disc buried in an infinite medium have also been considered as in Collins (1962), Keer (1965), Hunter and Gamblen (1974), and Selvadurai (1976, 1980). On solutions pertaining to a disc embedded at a finite depth in a half-space, however, only the axisymmetric cases of torsion and axial translation have been investigated (Pak and Saphores, 1991; Pak and Gobert, 1990).

This paper is concerned with the determination of stresses and displacements in the interior of a half-space when an embedded rigid disc is forced to rotate about a horizontal centroidal axis. Within the framework of linear elasticity, the asymmetric mixed boundary value problem is reduced to a set of dual integral equations. Under appropriate limiting conditions, the analytical formulation is shown to encompass the corresponding full-space and surface-disc problems as degenerate cases. Through this exposition, a severe boundary-layer effect on the contact load distribution is revealed. For practical applications, the rocking stiffness of the embedded disc and the response of the medium as a function of the embedment depth are included.

2. MATHEMATICAL FORMULATION

Consider a rigid disc of radius a located at a depth s in a homogeneous, isotropic, linearly elastic half-space. As shown in Fig. 1, the disc is assumed to be undergoing a rigid body rotation Ω about the y-axis due to a set of loads equivalent to a moment. In cylindrical coordinates, a relaxed treatment of this mixed boundary value problem can be stated in terms of the components of the displacement field $\mathbf{u}$ and the stress field $\boldsymbol{\sigma}$ as follows:

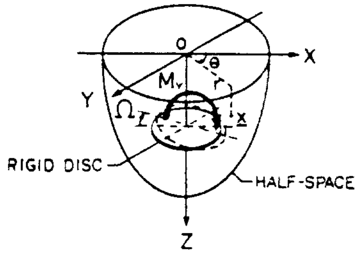

Fig. 1. Rocking rotation of a rigid disc in a half-space.

$$u_z(r,\theta,s) = \Omega r\cos\theta, \quad r < a, \quad \theta\in[0,2\pi] \tag{1}$$

$$u_z(r,\theta,s^-) = u_z(r,\theta,s^+), \quad r > a, \quad \theta\in[0,2\pi] \tag{2}$$

$$u_r(r,\theta,s^-) = u_r(r,\theta,s^+), \quad r\in[0,\infty), \quad \theta\in[0,2\pi] \tag{3}$$

$$u_\theta(r,\theta,s^-) = u_\theta(r,\theta,s^+), \quad r\in[0,\infty), \quad \theta\in[0,2\pi] \tag{4}$$

$$\sigma_{zz}(r,\theta,0) = \sigma_{zr}(r,\theta,0) = \sigma_{z\theta}(r,\theta,0) = 0, \quad r\in[0,\infty), \quad \theta\in[0,2\pi] \tag{5}$$

$$\sigma_{zr}(r,\theta,s^-) = \sigma_{zr}(r,\theta,s^+), \quad r\in[0,\infty), \quad \theta\in[0,2\pi] \tag{6}$$

$$\sigma_{z\theta}(r,\theta,s^-) = \sigma_{z\theta}(r,\theta,s^+), \quad r\in[0,\infty), \quad \theta\in[0,2\pi] \tag{7}$$

$$\sigma_{zz}(r,\theta,s^-)-\sigma_{zz}(r,\theta,s^+) = R(r,\theta;s), \quad r < a, \quad \theta\in[0,2\pi] \tag{8}$$

$$\sigma_{zz}(r,\theta,s^-) = \sigma_{zz}(r,\theta,s^+), \quad r > a, \quad \theta\in[0,2\pi]. \tag{9}$$

Here, $R(r,\theta;s)$ denotes the unknown resultant normal contact stress distribution acting on the disc at $z = s$. As an unbounded region is under consideration, the foregoing requirements must be adjoined by the regularity conditions at infinity that

$$\sigma \to 0 \quad \text{as} \quad \sqrt{r^2+z^2} \to \infty. \tag{10}$$

To obtain a mathematical formulation for the problem under consideration, it is useful to determine first the displacement response of a half-space under an arbitrary vertical body force field $R(r,\theta;s)$ distributed on the $z = s$ plane. For this purpose, the method of potentials by Muki (1960) is useful. As a specific reduction of the general Boussinesq–Somigliano–Galerkin solution, the method entails representing the solutions to the displacement equation of equilibrium with zero body-force field in terms of a biharmonic field Φ and a harmonic field Ψ, i.e.,

$$\nabla^4\Phi = 0, \quad \nabla^2\Psi = 0 \tag{11}$$

where

$$\nabla^4 = \nabla^2\nabla^2, \quad \nabla^2 = \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2} \tag{12}$$

in cylindrical coordinates. In terms of the two potentials, the components of the displacement vector and the stress tensor can be expressed as

$$2\mu u_r = -\frac{\partial^2\Phi}{\partial r\,\partial z}+\frac{2}{r}\frac{\partial\Psi}{\partial\theta}, \quad 2\mu u_\theta = -\frac{1}{r}\frac{\partial^2\Phi}{\partial\theta\,\partial z}-2\frac{\partial\Psi}{\partial r}, \quad 2\mu u_z = 2(1-\nu)\nabla^2\Phi-\frac{\partial^2\Phi}{\partial z^2} \tag{13}$$

and

$$\sigma_{rr} = \frac{\partial}{\partial z}\left(\nu\nabla^2 - \frac{\partial}{\partial r^2}\right)\Phi + 2\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\Psi \tag{14}$$

$$\sigma_{\theta\theta} = \frac{\partial}{\partial z}\left(\nu\nabla^2 - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\Phi - 2\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\Psi \tag{15}$$

$$\sigma_{zz} = \frac{\partial}{\partial z}\left((2-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right)\Phi \tag{16}$$

$$\sigma_{zr} = \frac{\partial}{\partial r}\left((1-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right)\Phi + \frac{1}{r}\frac{\partial^2\Psi}{\partial\theta\,\partial z} \tag{17}$$

$$\sigma_{z\theta} = \frac{1}{r}\frac{\partial}{\partial\theta}\left((1-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right)\Phi - \frac{\partial^2\Psi}{\partial r\,\partial z} \tag{18}$$

$$\sigma_{r\theta} = \frac{1}{r}\frac{\partial^2}{\partial\theta\,\partial z}\left[\frac{1}{r} - \frac{\partial}{\partial r}\right]\Phi - \left[2\frac{\partial^2}{\partial r^2} - \frac{\partial^2}{\partial z^2}\right]\Psi \tag{19}$$

where μ and ν are the shear modulus and the Poisson's ratio of the medium, respectively. By virtue of the completeness of the set of eigenfunctions $\{e^{im\theta}\}_{m=-\infty}^{\infty}$ for the class of solutions under consideration, one may write

$$\Phi(r,\theta,z;s) = \sum_{m=-\infty}^{\infty} \Phi_m(r,z;s)e^{im\theta} \tag{20}$$

$$\Psi(r,\theta,z;s) = \sum_{m=-\infty}^{\infty} \Psi_m(r,z;s)e^{im\theta} \tag{21}$$

$$R(r,\theta;s) = \sum_{m=-\infty}^{\infty} R_m(r;s)e^{im\theta}, \quad \text{etc.} \tag{22}$$

Owing to the orthogonality of $\{e^{im\theta}\}$, it is evident that (11) implies

$$\nabla_m^2\nabla_m^2\Phi_m = 0, \quad \nabla_m^2\Psi_m = 0, \quad \forall m \tag{23}$$

$$\nabla_m^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{m^2}{r^2} + \frac{\partial^2}{\partial z^2}. \tag{24}$$

In view of the differential operators and boundary conditions involved, it is natural to appeal to the theory of Hankel transforms in the solution of the foregoing equations. To this end, one defines the mth order Hankel transform as

$$\bar{f}^m(\xi) = \int_0^\infty f(r)rJ_m(\xi r)\,\mathrm{d}r \tag{25}$$

where J_m is the Bessel Function of the first kind of order m. With the aid of (25) with respect to the radial coordinate, the partial differential equations in (23) can be reduced to two ordinary differential equations for each pair of $\bar{\Phi}_m^m$ and $\bar{\Psi}_m^m$.

On account of the regularity conditions at infinity in the problem under consideration, the general solutions can be expressed as

$$\tilde{\Phi}_m^m(\xi, z; s) = \begin{cases} (A_m^{\rm I}(\xi) + zB_m^{\rm I}(\xi))\,{\rm e}^{-\xi z} + (C_m^{\rm I}(\xi) + zD_m^{\rm I}(\xi))\,{\rm e}^{\xi z}, & z < s \\ (A_m^{\rm II}(\xi) + zB_m^{\rm II}(\xi))\,{\rm e}^{-\xi z}, & z > s \end{cases} \quad (26)$$

$$\tilde{\Psi}_m^m(\xi, z; s) = \begin{cases} E_m^{\rm I}(\xi)\,{\rm e}^{-\xi z} + F_m^{\rm I}(\xi)\,{\rm e}^{\xi z}, & z < s \\ E_m^{\rm II}(\xi)\,{\rm e}^{-\xi z}, & z > s \end{cases}. \quad (27)$$

By virtue of (2)–(8), the functions $A_m^{\rm I}(\xi), \ldots, E_m^{\rm II}(\xi)$, and thus the full solution for the auxiliary problem, can be determined explicitly in terms of the transform of the Fourier component $R_m(r; s)$ of the contact load distribution. From (1), one may further deduce that

$$R_1 = R_{-1}; \quad R_m = 0, \quad m \notin \{1, -1\}. \quad (28)$$

Accordingly, the vertical component of the displacement field can be expressed as

$$u_z(r, \theta, z) = 2u_{z_1}(r, z)\cos\theta \quad (29)$$

where

$$u_{z_1}(r, z) = \int_0^\infty \Omega_2(\xi, z; s)\frac{\xi Z_1(\xi; s)}{\mu} J_1(\xi r)\,{\rm d}\xi \quad (30)$$

$$Z_1(\xi; s) = \tilde{R}_1^1(\xi; s) \quad (31)$$

$$\Omega_2(\xi, z; s) = \frac{1}{8(1-\nu)\xi}\begin{pmatrix} (3 - 4\nu + \xi|z-s|)\,{\rm e}^{-|z-s|\xi} + \\ (5 - 12\nu + 8\nu^2 + (3-4\nu)(z+s)\xi + 2zs\xi^2)\,{\rm e}^{-(z+s)\xi} \end{pmatrix}. \quad (32)$$

With the aid of (29) and (30), it can be shown that the remaining two conditions (1) and (9) of the mixed boundary value problem are equivalent to the dual integral equations

$$\int_0^\infty \Omega_2(\xi, s; s)\frac{\xi Z_1(\xi; s)}{\mu} J_1(\xi r)\,{\rm d}\xi = \frac{\Omega r}{2}, \quad r < a \quad (33a)$$

and

$$\int_0^\infty \xi Z_1(\xi; s) J_1(\xi r)\,{\rm d}\xi = 0, \quad r > a \quad (33b)$$

where

$$\Omega_2(\xi, s; s) = \frac{3-4\nu}{8(1-\nu)\xi}\left\{1 + \left(\frac{5 - 12\nu + 8\nu^2}{3-4\nu} + 2\xi s + \frac{2\xi^2 s^2}{3-4\nu}\right){\rm e}^{-2\xi s}\right\} \quad (34)$$

with the property that

$$\lim_{\xi \to \infty} \xi\Omega_2(\xi, s; s) = \frac{(3-4\nu)}{8(1-\nu)}. \quad (35)$$

By setting $s = 0$ in (33), one finds that the governing equations become

$$\int_0^\infty \Omega_2(\xi, 0; 0)\frac{\xi Z_1(\xi; 0)}{\mu} J_1(\xi r)\,{\rm d}\xi = \frac{\Omega r}{2}, \quad r < a \quad (36a)$$

and

$$\int_0^\infty \xi Z_1(\xi;0)J_1(\xi r)\,d\xi = 0, \quad r > a \tag{36b}$$

with

$$\Omega_2(\xi,0;0) = \frac{(1-\nu)}{\xi}. \tag{37}$$

Upon interpreting $Z_1(\xi;0)$ as the transform of the contact normal stress distribution underneath the disc, the formulation in (36) is identical to the one for a rigid punch in smooth contact with the surface of a half-space.

3. REDUCTION OF DUAL INTEGRAL EQUATIONS

With the aid of Sonine's integrals as in Noble (1963), eqn (33) can be transformed to

$$\int_0^\infty \frac{1}{\sqrt{\xi}}(1+H_s(\xi))Z(\xi;s)J_{1/2}(r\xi)\,d\xi = \frac{8(1-\nu)}{3-4\nu}\Omega\sqrt{\frac{2r}{\pi}}, \quad r < a \tag{38a}$$

and

$$\int_0^\infty \frac{1}{\sqrt{\xi}}Z(\xi;s)J_{1/2}(r\xi)\,d\xi = 0, \quad r > a, \tag{38b}$$

where

$$Z(\xi;s) = \frac{\xi Z_1(\xi;s)}{\mu}, \tag{39}$$

$$H_s(\xi) = \left(\frac{5-12\nu+8\nu^2}{3-4\nu} + 2\xi s + \frac{2\xi^2 s^2}{3-4\nu}\right)e^{-2\xi s}. \tag{40}$$

For further reduction, it is useful to express

$$\int_0^\infty \frac{1}{\sqrt{\xi}}Z(\xi;s)J_{1/2}(r\xi)\,d\xi = \sqrt{\frac{2}{\pi r}}\eta_s(r), \quad r < a \tag{41}$$

where $\eta_s(r)$ is an unknown function. By taking $\eta_s(r) = 0$ for $r > a$, it is evident that the foregoing representation of Z would satisfy (38b) identically. From the Hankel inversion theorem, it also follows that (41) yields

$$Z(\xi;s) = \sqrt{\frac{2}{\pi}}\xi^{3/2}\int_0^a r^{1/2}\eta_s(r)J_{1/2}(\xi r)\,dr. \tag{42}$$

Upon substituting (42) into (38a), one can readily verify that the governing dual integral equations are equivalent to a Fredholm integral equation of the second kind

$$\eta_s(r) + \int_0^a K_s(r,\rho)\eta_s(\rho)\,d\rho = f_s(r) \tag{43}$$

where

$$K_s(r,\rho) = \sqrt{r\rho}\int_0^\infty \xi H_s(\xi) J_{1/2}(r\xi) J_{1/2}(\rho\xi)\,\mathrm{d}\xi \tag{44}$$

$$f_s(r) = \frac{8(1-\nu)}{3-4\nu}\Omega r. \tag{45}$$

For the present problem, the kernel of the integral equation can be evaluated in closed form as

$$K_s(r,\rho) = k(r-\rho) - k(r+\rho) \tag{46}$$

where

$$k(x) = \frac{2}{\pi}\left[\frac{(5-12\nu+8\nu^2)}{(3-4\nu)}\frac{s}{(x^2+4s^2)} + \frac{(4s^3-sx^2)}{(x^2+4s^2)^2} + \frac{2}{(3-4\nu)}\frac{(8s^5-6s^3x^2)}{(x^2+4s^2)^3}\right]. \tag{47}$$

For the case of $s = 0$, the analysis of the dual integral equations is considerably simpler. In fact, by the same transformation in (41), the solution to (36) can be written as

$$\eta_0(r) = \frac{\Omega r}{1-\nu}. \tag{48}$$

4. SOLUTION OF FREDHOLM INTEGRAL EQUATION

Before proceeding to the treatment of the general embedment problem, it is relevant to examine some special cases for which closed-form solutions are available.

Case A: $s \to \infty$

For the case of a disc buried in an infinite medium, the kernel (46) degenerates to zero and the solution can be immediately written as

$$\eta_\infty(r) = \frac{8(1-\nu)}{3-4\nu}\Omega r. \tag{49}$$

Case B. $s = 0$

As indicated in the preceding section, the solution to the case of a disc acting on a half-space is

$$\eta_0(r) = \frac{\Omega r}{1-\nu}. \tag{50}$$

An interesting but peculiar result though is the solution obtained in the limit of $s \to 0$ in the general embedment problem. To this end, it is useful to recognize that

$$\frac{s}{x^2+4s^2} \to \frac{\pi}{2}\delta(x),\quad \frac{sx^2}{(x^2+4s^2)^2} \to \frac{\pi}{4}\delta(x),\quad \frac{s^3}{(x^2+4s^2)^2} \to \frac{\pi}{16}\delta(x),$$
$$\frac{s^3x^2}{(x^2+4s^2)^3} \to \frac{\pi}{64}\delta(x),\quad \frac{s^5}{(x^2+4s^2)^3} \to \frac{3\pi}{256}\delta(x),\quad \text{as}\quad s \to 0 \tag{51}$$

where $\delta(x)$ stands for a symmetric Dirac delta-function in the theory of distribution. With the aid of (51), one can readily see that

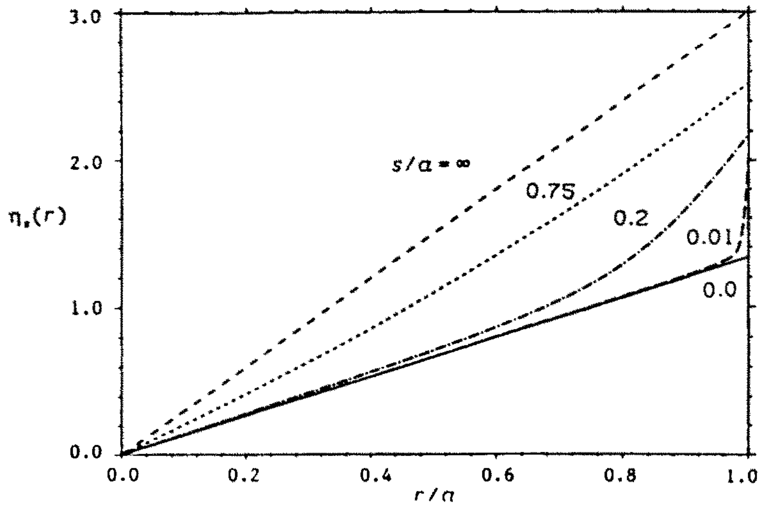

Fig. 2. Typical solutions for η_s ($\nu = 0.25$).

$$\lim_{s\to 0} K_s(r,\rho) = \frac{5-12\nu+8\nu^2}{3-4\nu}\{\delta(r-\rho)-\delta(r+\rho)\} \tag{52}$$

and the Fredholm integral equation yields

$$\eta_0+(r) = \begin{Bmatrix} \dfrac{\Omega r}{1-\nu}, & r<a \\ \dfrac{16(1-\nu)\Omega a}{11-20\nu+8\nu^2}, & r\to a \end{Bmatrix} \tag{53}$$

which exhibits a finite jump at the edge of the disc.

Case C: $0 < s < \infty$

For finite embedments, numerical solutions of the integral equation can be obtained by standard quadrature methods. Typical solutions for η_s are shown in Fig. 2. As portended by (53), it can be seen that there is a severe boundary layer in the solution at the edge region of the disc as the embedment depth tends to zero. The implications of the jump of η_s and the emergence of the boundary layer in such a limit process will be explored in a later section.

5. ROCKING STIFFNESS

The moment M_y required to sustain the rotation Ω is defined by

$$M_y = \int_0^a \int_0^{2\pi} R(r,\theta;s)r^2\cos\theta \, d\theta \, dr. \tag{54}$$

By virtue of (28), (31), (39), (42), and the identity

$$\int_0^\infty J_2(\xi a)\sin(\xi r)\,d\xi = \frac{2r}{a^2}, \quad r<a, \tag{55}$$

(54) can be simplified to

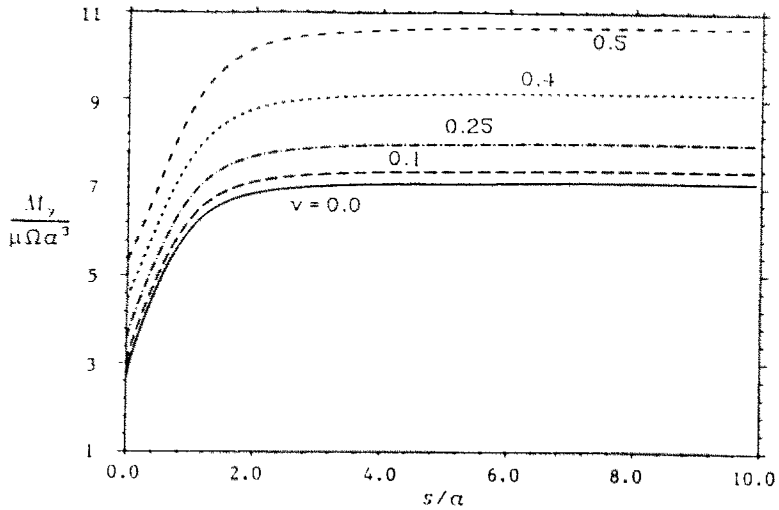

Fig. 3. Rotational stiffness of disc as a function of embedment.

$$M_y = 8\mu \int_0^a r\eta_s(r)\,\mathrm{d}r \tag{56}$$

which can be evaluated directly in terms of the solution of the Fredholm integral equation. For $s \to \infty$, (56) yields

$$M_y = \frac{64(1-\nu)\mu a^3\Omega}{3(3-4\nu)} \tag{57}$$

as in Selvadurai (1980). For $s = 0$, one finds

$$M_y = \frac{8\mu\Omega a^3}{3(1-\nu)} \tag{58}$$

which is in agreement with Abramov (1939) and Bycroft (1956). The general rocking stiffness, which is defined as

$$K_{RR} = \frac{M_y}{\mu\Omega a^3} \tag{59}$$

as a function of s is illustrated in Fig. 3. From the display, it is evident that the stiffness increases with the depth of embedment as well as Poisson's ratio. As s reaches the value of $4a$, however, the stiffness appropriate to $s \to \infty$ is virtually attained.

6. CONTACT LOAD DISTRIBUTION

By virtue of the Hankel inversion theorem, eqn (42) yields the distribution $R_1(r;s)$ pertaining to the contact load on the disc as

$$R_1(r;s) = \mu\sqrt{\frac{2}{\pi}}\int_0^\infty \xi^{3/2}\int_0^a \rho^{1/2}\eta_s(\rho)J_{1/2}(\xi\rho)\,\mathrm{d}\rho\, J_1(\xi r)\,\mathrm{d}\xi. \tag{60}$$

With the aid of the identity

$$\int_0^\infty \sin(\xi\rho)J_0(\xi r)\,\mathrm{d}\xi = \begin{Bmatrix} 0, & \rho < r \\ \dfrac{1}{\sqrt{\rho^2 - r^2}}, & \rho > r \end{Bmatrix} \tag{61}$$

one finds the resultant contact load distribution

$$R(r,\theta;s) = -\frac{4\mu\cos\theta}{\pi}\frac{\mathrm{d}}{\mathrm{d}r}\int_r^a \frac{\eta_s(\rho)}{\sqrt{\rho^2-r^2}}\,\mathrm{d}\rho. \tag{62}$$

With an integration by parts, eqn (62) can be reduced to

$$R(r,\theta;s) = \frac{4\mu}{\pi}\left[\frac{a\eta_s(a)}{r\sqrt{a^2-r^2}} - \frac{1}{r}\int_r^a \frac{\rho\eta_s'(\rho)\,\mathrm{d}\rho}{\sqrt{\rho^2-r^2}}\right]\cos\theta. \tag{63}$$

For $s \to \infty$, eqn (63) yields

$$R(r,\theta;\infty) = \frac{32(1-\nu)\mu}{\pi(3-4\nu)}\frac{\Omega r\cos\theta}{\sqrt{a^2-r^2}}, \quad r \in [0,a). \tag{64}$$

For $s = 0$, one finds

$$R(r,\theta;0) = \frac{4\mu}{\pi(1-\nu)}\frac{\Omega r\cos\theta}{\sqrt{a^2-r^2}}, \quad r \in [0,a). \tag{65}$$

In view of the difference between the solutions (53) and (50), however, one can anticipate that the contact load distribution for $s \to 0+$ is apt to deviate from the one described in (65). Indeed, it follows from (63) that while

$$\lim_{s\to 0}\frac{R(r,\theta;s)}{R(r,\theta;0)} = 1, \quad r\in[0,a), \quad \theta\in[0,2\pi], \tag{66}$$

$$\lim_{s\to 0}\lim_{r\to a}\frac{R(r,\theta;s)}{R(r,\theta;0)} = \frac{\eta_{0^+}(a)}{\eta_0(a)} = \frac{16(1-\nu)^2}{11-20\nu+8\nu^2}. \tag{67}$$

The general variation of the singular contact load distribution as a function of s is illustrated in Fig. 4.

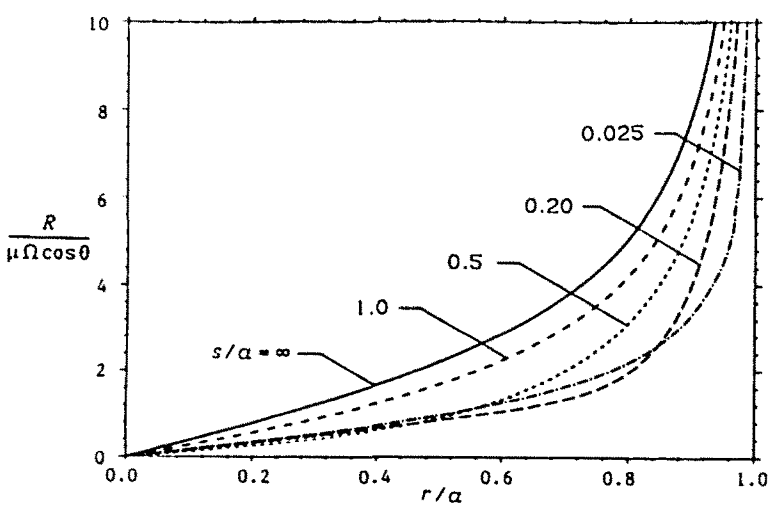

Fig. 4. Contact load distributions on disc ($\nu = 0.25$).

7. STRESS AND DISPLACEMENT

By virtue of (26), (27) and their relations to $Z_1(\xi; s)$, the stresses in the medium can also be expressed directly in terms of the function η_s. For instance, with the aid of (16), one finds

$$\sigma_{zz} = 2\cos\theta \int_0^\infty \xi\Omega_4(\xi, z; s) Z_1(\xi; s) J_1(\xi r)\, d\xi \tag{68}$$

where

$$\Omega_4(\xi, z; s) = \frac{1}{4(1-\nu)} \left\{ \begin{matrix} \operatorname{sgn}(z-s)(2(1-\nu)+d_1\xi)\, e^{-d_1\xi} \\ +(2(1-\nu)+s\xi+(3-4\nu)z\xi-2sz\xi^2)\, e^{-d_2\xi} \end{matrix} \right\}, \tag{69}$$

$$\operatorname{sgn}(z-s) = \left\{ \begin{matrix} +1, & z > s \\ -1, & z < s \end{matrix} \right\}. \tag{70}$$

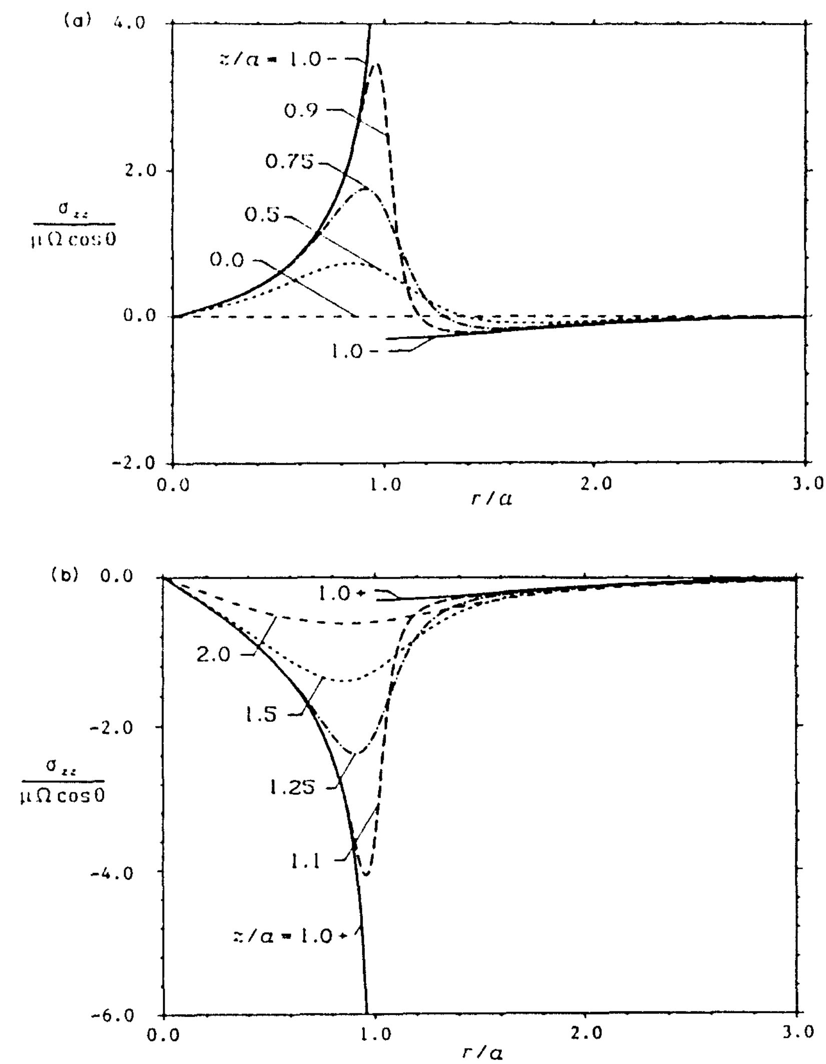

Fig. 5. Distribution of σ_{zz} ($s/a = 1$, $\nu = 0.25$). (a) $z < s$. (b) $z > s$.

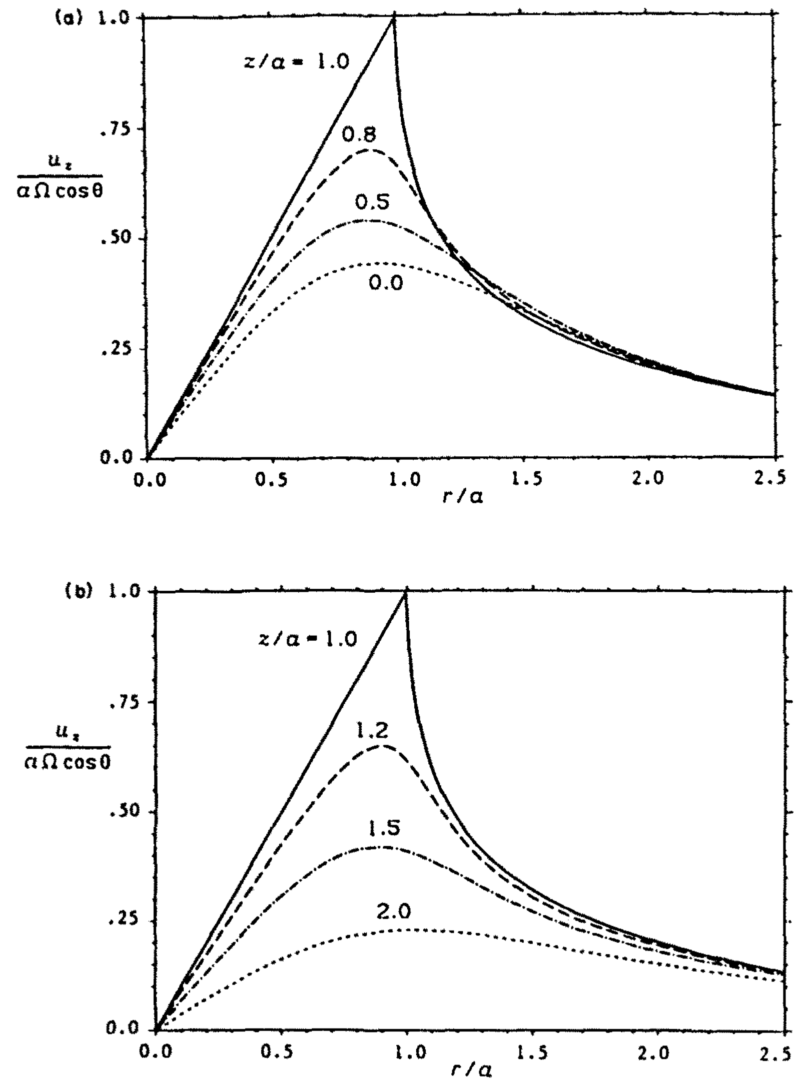

Fig. 6. Vertical displacement u_z ($s/a = 1$, $\nu = 0.25$). (a) $z < s$. (b) $z > s$.

$$d_1 = |z-s|, \quad d_2 = z+s. \tag{71}$$

In terms of η_s and the notation

$$S(\rho, r, d; n, p) = \int_0^\infty \xi^n \, e^{-\xi d} \sin(\xi\rho) J_p(\xi r) \, d\xi, \tag{72}$$

eqn (68) can be written as

$$\sigma_{zz}(r,\theta,z) = \frac{\mu\cos\theta}{\pi(1-\nu)} \int_0^a \left\{ \begin{array}{c} \operatorname{sgn}(z-s) 2(1-\nu) S(\rho, r, d_1; 1, 1) \\ + d_1 S(\rho, r, d_1; 2, 1) \\ + 2(1-\nu) S(\rho, r, d_2; 1, 1) \\ + ((3-4\nu)z+s) S(\rho, r, d_2; 2, 1) \\ -2zs S(\rho, r, d_2; 3, 1) \end{array} \right\} \eta_s(\rho) \, d\rho. \tag{73}$$

As the functions S required in (73) can be evaluated in closed form, σ_{zz} can be computed by ordinary quadrature. Typical distributions of σ_{zz} in the half-space are displayed in Fig. 5.

As indicated in (29), the vertical displacement field can be expressed as

$$u_z(r,\theta,z) = 2\cos\theta \int_0^{\infty} \Omega_2(z,s,\xi)\frac{\xi Z_1(\xi;s)}{\mu} J_1(\xi r)\,\mathrm{d}\xi. \tag{74}$$

By virtue of (42), one finds

$$u_z(r,\theta,z) = \frac{\cos\theta}{2\pi(1-\nu)}\int_0^a \left\{ \begin{array}{c} (3-4\nu)S(\rho,r,d_1;0,1) \\ +d_1 S(\rho,r,d_1;1,1) \\ +(5-12\nu+\nu^2)S(\rho,r,d_2;0,1) \\ +(3-4\nu)d_2 S(\rho,r,d_2;1,1) \\ +2zsS(\rho,r,d_2;2,1) \end{array} \right\} \eta_s(\rho)\,\mathrm{d}\rho \tag{75}$$

which can also be evaluated numerically in a straightforward manner. Some typical results are given in Fig. 6.

8. CONCLUSION

In this paper, an analytical treatment is presented for the determination of the response of an elastic half-space under the action of an embedded rigid disc rotating about a horizontal axis. With the aid of Hankel transforms, a mathematical formulation is developed for the mixed boundary value problem in the form of dual integral equations. Illustrative results on the influence of embedment on the moment-rotation relationship and the stress field, as well as the displacement field, are included. In addition to providing a unified view of previous works on the cases of surface and infinite embedment, the present treatment reveals a severe boundary-layer phenomenon in the contact load distribution which is apt to be of interest to this class of problems in general.

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